Output Feedback Stabilization for a Class of Lipschitz Nonlinear Systems*

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SUMMARY In this letter, we provide a solution to the stabilization problem of a class of Lipschitz nonlinear systems by output feedback. Via the newly proposed nonlinearity characterization function (NCF) concept, we propose an effective method in designing an output feedback controller. Under the suggested sufficient condition which is derived by using the NCF, the proposed control scheme achieves the global exponential stabilization.

key words: Lipschitz nonlinear systems, output feedback controller, global exponential stabilization

1. Introduction

In recent years, there have been increasing attention for the problem of global stabilization of nonlinear systems by output feedback. Unlike in the case of linear systems, the separation principle generally does not hold for nonlinear systems. This is one of reasons why the problem is more difficult. In [6], it was studied through counter examples that some extra growth conditions on the unmeasurable states of the system are usually required for the global stabilization of nonlinear systems via output feedback.

Recent research has focused on considering a selective class of nonlinear systems by placing some structural conditions on the nonlinearities in order to solve the output feedback stabilization problem. The nonlinear systems whose nonlinearity is in a triangular form is considered in [4], [9]. Observer design techniques were extensively studied in [1], [7], [8], [10]. In [8], the existence of a stable observer for Lipschitz nonlinear systems was addressed and a sufficient condition was derived on the Lipschitz constant. Some of the results of [8] were corrected in [1]. In [7], another technique to design a stable observer was studied. Specifically, the method of designing an observer gain was suggested. Later on, the result of [7] was extended to a reduced-observer design case in [10].

The systems with Lipschitz nonlinearities are common in practical applications. In this letter, we provide a solution to the output feedback stabilization problem for Lipschitz nonlinear systems under a newly suggested condition. Via the nonlinearity characterization function (NCF) concept which is formulated using the inequality in [3], the system under consideration is not restricted to a triangular form. Moreover, this NCF concept can be effectively used as a tool in designing the output feedback controller. In [3], the proposed controller is an linear output feedback controller which often leads to too conservative controller designs. In this letter, we show that the proposed nonlinear output feedback controller achieves the global exponential stabilization based on the global nonlinear separation principle, which results in the more flexible and practical controller design.

2. Preliminaries

We consider a class of single-input single-output nonlinear systems given by
\begin{align}
\dot{x} &= Ax + Bu + \delta(t, x, u) \\
y &= Cx
\end{align}
where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}\) and \(y \in \mathbb{R}\) are the input and the output of the system, respectively. The system matrices are
\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix},
C = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
The nonlinear term is given by \(\delta(t, x, u) = [\delta_1(t, x, u), \cdots, \delta_n(t, x, u)]^T\). The mappings \(\delta_i(t, x, u) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, i = 1, \cdots, n\), are continuous and satisfy the following assumption:

Assumption 1. (\(\epsilon\)-dependent global Lipschitz-like condition) There exists a function \(\gamma(\epsilon) \geq 0\) such that for \(\epsilon > 0\)
\[
\sum_{i=1}^{n} e^{\epsilon^{-1}|\delta_i(t, x, u) - \delta_i(t, \hat{x}, u)|} \leq \gamma(\epsilon) \sum_{i=1}^{n} e^{\epsilon^{-1}|x_i - \hat{x}_i|}
\]

Definition 1. The function \(\gamma(\epsilon)\) of Assumption 1 is called a nonlinearity characterization function (NCF) of the system (1), (2).

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We note that the NCF $\gamma(\epsilon)$ can characterize the nonlinearities of the system (1), (2). In the following, we provide three examples showing various types of NCFs.

**Example A:** (Increasing-type NCF) Consider a simple inverted pendulum model which describes the mechanics of the limb movement

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \frac{mg}{J} \sin x_1 \\
y &= x_1
\end{align*}
$$

(4)

For this system, note that $|\dot{x}_2(t, x, u) - \dot{x}_2(t, \hat{x}, u)| \leq \frac{mg}{J} |\sin x_1 - \sin \hat{x}_1|$. In view of Assumption 1, we have $\epsilon^2 \frac{mg}{J} |\sin x_1 - \sin \hat{x}_1| \leq \gamma(\epsilon) |x_1 - \hat{x}_1|$. Thus, the NCF can be $\gamma(\epsilon) = \frac{mg}{J} \epsilon^2$.

**Example B:** (Decreasing-type NCF) Consider the following feedforward system

$$
\begin{align*}
\dot{x}_1 &= x_2 + \frac{x_3}{8} \sin u \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u \\
y &= x_1
\end{align*}
$$

(5)

In view of Assumption 1, we have $\frac{1}{8} |x_3 \sin u - \hat{x}_3 \sin u| \leq \gamma(\epsilon) |x_3 - \hat{x}_3|$. Thus, the NCF can be $\gamma(\epsilon) = 1/(8\epsilon^2)$.

**Example C:** (Parabolic-type NCF) Consider the following system

$$
\begin{align*}
\dot{x}_1 &= x_2 + \frac{x_3}{8} \sin t \\
\dot{x}_2 &= x_3 + \frac{x_1}{2} \sin u \\
\dot{x}_3 &= u \\
y &= x_1
\end{align*}
$$

(6)

In view of Assumption 1, we have $\frac{1}{8} |x_2 \sin t - \hat{x}_2 \sin t| + \epsilon \frac{1}{8} |x_1 \sin u - \hat{x}_1 \sin u| \leq \gamma(\epsilon) |x_1 - \hat{x}_1| + \epsilon |x_2 - \hat{x}_2|$. Thus, the NCF can be $\gamma(\epsilon) = \max(1/(8\epsilon), \epsilon^2/2)$.

3. **Main Result**

First, the proposed nonlinear output feedback controller is given by

$$
\begin{align*}
\dot{u} &= K(\epsilon_1) \dot{x} \\
\dot{x} &= A(\epsilon_1) x + B(\epsilon_1) y - C(\dot{x}) + \delta(t, \dot{x}, u)
\end{align*}
$$

(7)

where $A(\epsilon_1) = \left[ \begin{array}{c} \delta_1(t, \dot{x}, u, \cdots, \delta_9(t, \dot{x}, u) \end{array} \right]^T$, $K(\epsilon_1) = \left[ \begin{array}{c} \epsilon_1 K \\ \epsilon_2 K \\ \cdots \\ \epsilon_9 K \end{array} \right]$, and $L(\epsilon_1) = \left[ \begin{array}{c} \epsilon_1 L \\ \epsilon_2 L \\ \cdots \\ \epsilon_9 L \end{array} \right]^T$ with $\epsilon_k, \epsilon_L > 0$.

**Controller design procedure:**

(i) Obtain the NCF $\gamma(\epsilon)$ of the considered system (1), (2).

(ii) Select $K = [k_1, \cdots, k_9]$ and $L = [l_1, \cdots, l_9]^T$ such that $A_K \hat{=} A + BK$ and $A_L \hat{=} A + LC$ are Hurwitz.

(iii) Calculate $\sigma_1 = 2 \sqrt{n} \|P_1\|$ and $\sigma_2 = 2 \sqrt{n} \|P_2\|$ where $A_K^2 P_K + P_K A_K = -I$ and $A_L^2 P_L + P_L A_L = -I$ ($\sigma_1$ and $\sigma_2$ are $\epsilon_k, \epsilon_L$-independent constants).

(iv) Select $\epsilon_K$ and $\epsilon_L$ such that $\epsilon_K^{-1} - \sigma_1 \gamma(\epsilon_K) > 0$ and $\epsilon_L^{-1} - \sigma_2 \gamma(\epsilon_L) > 0$ where $\gamma(\epsilon_K) = \gamma(\epsilon)|_{\epsilon=\epsilon_K}$ and $\gamma(\epsilon_L) = \gamma(\epsilon)|_{\epsilon=\epsilon_L}$.

**Theorem 1.** Suppose that (a) Assumption 1 holds, (b) there exist $K$ and $\epsilon_K$ such that $A_K$ is Hurwitz and $\epsilon_K^{-1} - \sigma_1 \gamma(\epsilon_K) > 0$, (c) there exist $L$ and $\epsilon_L$ such that $A_L$ is Hurwitz and $\epsilon_L^{-1} - \sigma_2 \gamma(\epsilon_L) > 0$. Then, with the nonlinear output feedback control scheme (7) and (8), the origin of the system (1), (2) is globally exponentially stable.

**Proof.** Define $e_i = x_i - \hat{x}_i, i = 1, \cdots, n$. By subtracting (8) from (1) and with the controller (7), the augmented closed-loop dynamics is

$$
\begin{align*}
\dot{e} &= A(\epsilon_1) e + \delta(t, x, u) - \delta(t, \dot{x}, u) \\
\dot{x} &= A(\epsilon_1) x + \delta(t, x, u) - BK(\epsilon_K) e
\end{align*}
$$

(9)

(10)

where $A_K(\epsilon_K) = A + BK(\epsilon_K)$ and $A_L(\epsilon_L) = A + LC(\epsilon_K)$. First, denote $E(\epsilon_L) = \text{diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_L]$. Then, we have the following equalities:

$$
\begin{align*}
e_i A_L(\epsilon_L) e &= E(\epsilon_L^{-1}) A_L E(\epsilon_L) \\
A_L^2(\epsilon_L) P_L(\epsilon_L) + P_L(\epsilon_L) A_L(\epsilon_L) &= -\epsilon_L^{-1} E(\epsilon_L^2) \\
P_L(\epsilon_L) &= E(\epsilon_L) P_L(\epsilon_L)
\end{align*}
$$

We set $V_o(e) = e^T P_L(\epsilon_L) e$ for (9). Then, along the trajectory of (9),

$$
\begin{align*}
V_o(e) &= -\epsilon_L^{-1} \|E(\epsilon_L)e\|^2 \\
&+ 2\epsilon_L^2 \|E(\epsilon_L)e\|[\delta(t, x, u) - \delta(t, \dot{x}, u)] \\
&\leq -\epsilon_L^{-1} \|E(\epsilon_L)e\|^2 + 2\|P_L(\epsilon_L)\|[\|E(\epsilon_L)e\|] \\
&\times [\|E(\epsilon_L)[\delta(t, x, u) - \delta(t, \dot{x}, u)]\|]_1
\end{align*}
$$

(11)

Here, from Assumption 1, we have $\|E(\epsilon_L)[\delta(t, x, u) - \delta(t, \dot{x}, u)]\| \leq \sqrt{n} \gamma(\epsilon_L) \|E(\epsilon_L)e\|$. Thus, we obtain

$$
V_o(e) \leq -N_o \|E(\epsilon_L)e\|^2, N_o = \epsilon_L^{-1} - \sigma_\gamma(\epsilon_L) > 0
$$

(12)

Second, with $E(\epsilon_K) = \text{diag}[\epsilon_1, \epsilon_2, \cdots, \epsilon_K]$, we have the similar equalities:

$$
\begin{align*}
\epsilon_K A_L(\epsilon_K) &= E(\epsilon_K^{-1}) A_L E(\epsilon_K) \\
A_L^2(\epsilon_K) P_L(\epsilon_K) + P_L(\epsilon_K) A_L(\epsilon_K) &= -\epsilon_K^{-1} E(\epsilon_K^2) \\
P_L(\epsilon_K) &= E(\epsilon_K) P_L(\epsilon_K)
\end{align*}
$$

Now, we set $V_o(x) = x^T P_K(\epsilon_K) x$ for (10). Then, along the trajectory of (10),

$$
\begin{align*}
V_o(x) &\leq -\epsilon_K^{-1} \|E(\epsilon_K)x\|^2 \\
&+ 2\epsilon_K^2 \|P_K(\epsilon_K)\|[\delta(t, x, u) - BK(\epsilon_K)e] \\
&\leq -\epsilon_K^{-1} \|E(\epsilon_K)x\|^2 \\
&+ 2\|P_K(\epsilon_K)\|[\|E(\epsilon_K)x\|][\|E(\epsilon_K)[\delta(t, x, u)]\|]_1 \\
&- 2\epsilon_K^2 \|E(\epsilon_K)P_L(\epsilon_K)BK(\epsilon_K)e\|
\end{align*}
$$

(13)

Here, from Assumption 1, we can deduce that $\|E(\epsilon_K)[\delta(t, x, u)]\| \leq \sqrt{n} \gamma(\epsilon_K) \|E(\epsilon_K)x\|$. Also, note that $E(\epsilon_K)BK(\epsilon_K)e = -\epsilon_K^{-1} BK \Delta E(\epsilon_K)e$ where $\Delta = \text{diag}[1/\epsilon_K, \cdots, (\epsilon_K\epsilon_L)^{-1}]$. Thus, we obtain
\[ \dot{V}(x) \leq -N_c \|E(e_K)\| + \rho \|E(e_K)\| \|E(e_L)\| \]

where \( N_c = e_K^{-1} - \sigma_1 \gamma(e_K) > 0 \) and \( \rho = e_L^{-1} \|P_c\| \|K\| \|A\| \).

Third, we set a composite Lyapunov function \( V(x) = V_o(e) + dV_c(x), d > 0 \). Then, along the trajectories of (9) and (10), we obtain
\[
\dot{V}(x) \leq -\Sigma^T M \Sigma
\]

where
\[
\Sigma = \left[ \begin{array}{c} \|E(e_K)\| \\ \|E(e_L)\| \end{array} \right], \quad M = \left[ \begin{array}{cc} N_c & -dp \\ -dp & dn_c \end{array} \right]
\]

Then, for any finite constant \( \epsilon > 0 \), the two design conditions are satisfied for
\[
\epsilon \leq \epsilon^* = -\epsilon < \epsilon^* < \infty.
\]

4. Illustrative Example

We reconsider Example C. This is another type of system in the sense that it does not satisfy either the above condition (a) or (b). First, recall that the NCF is obtained as \( \gamma(e) = \max\{\epsilon e, \epsilon\} \). The \( \gamma(e) \) is minimized when \( \epsilon = 0.5 \). Thus, in this case, the valid range for scaling-factors is mid-range, centering at \( \epsilon_K = \epsilon_L = 0.5 \). For the design of a controller, we select \( K = [-3.3750, -6.7500, -4.5000] \). Then, at \( \epsilon_K = 0.5 \), we have
\[
\gamma(e) = \max\{\epsilon e, \epsilon\} \quad \text{for the nonlinear observer, we select } L = [-6, -12, -8]^T.
\]

Then, at \( \epsilon_L = 0.5 \), we have
\[
\gamma(e) = \max\{\epsilon e, \epsilon\} \quad \text{for the nonlinear observer, the nonlinear terms are set as } L = [\epsilon_K, \epsilon_L, 0]^T.
\]

This completes the design of a nonlinear output feedback controller. The simulation result is shown in Fig. 1.

5. Conclusions

In this letter, we have proposed an output feedback control scheme to globally exponentially stabilize a class of Lipschitz nonlinear systems. The newly proposed NCF concept is utilized as a tool to design an output feedback controller in a systematic procedure. In particular, the gains of controller and observer can be designed separately based on the conditions given by the NCF. Moreover, with the NCF, we have identified two classes of nonlinear systems which can be stabilized by the proposed scheme for any finite Lipschitz constant.

References


