On stability of linear time-delay systems with multiple delays

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In this article, delay-dependent stability criteria for linear retarded and neutral systems with multiple delays are proposed by employing the Lyapunov-Krasovskii functional approach and integral inequality. By the \( N \)-segmentation of delay length, we obtain more relaxed results on the delay bounds which guarantee the asymptotic stability of the linear retarded and neutral systems with multiple delays. Simulation results show that the stability boundary calculated by the proposed stability criteria are very near to the real stability boundary of the example systems. Moreover, it is shown that the proposed stability criteria are less conservative than several other existing criteria.

Keywords: LMI; time-delay system; integral inequality; \( N \)-segmentation of delay length

1. Introduction

The stability of time-delay systems has been investigated in the last several decades. Stability criteria for linear time-delay systems can be divided into two categories, delay-independent stability criteria and delay-dependent stability criteria, by whether the criteria include the information about delays or not. In addition, linear time-delay systems can be divided into two categories by the number of delays, single-delay systems which have only a single time-delay constant and multiple-delay systems which have at least two time-delay constants.

The delay-independent stability criteria have been studied by many researchers (Brierley, Chiasson, and Zak 1982; Fridman and Shaked 2003; He, Wu, She, and Liu 2004; Mori, Fukuma, and Kuwahara 1981; Mori, and Kokame 1989; Su, Fung, and Tseng 1994; Xu 1995). It is well-known that delay-independent stability criteria are more conservative than delay-dependent stability criteria (Richard 2003). Many delay-dependent stability criteria (Kolmanovskii and Richard 1999; Li and Souza 1997; Moon, Park, Kwon, and Le 2001; Niculescu, Neto, Dion, and Dugard 1995; Park 1999) have been obtained by employing the Lyapunov functional approach (Hale and Lunel 1993; Kolmanovskii and Nosav 1986) and these criteria improved the conservative delay-independent stability criteria. Although these delay-dependent stability criteria are less conservative than the delay-independent stability criteria, they are still conservative. However, in recent years, Han, Gu, and Yu (2003) and Gouaisbaut and Peaucelle (2006a; 2006b) significantly improved the delay-dependent stability criteria for linear time-delay systems with single-delay. Han et al. (2003) developed the stability criterion based on the Gu’s discretised Lyapunov functional approach in Gu (1997). Gouaisbaut and Peaucelle (2006a) introduced discretisation of the delay in \( N \) internals of the same length to build a new Lyapunov-Krasovskii functional that reduces the conservatism. Gouaisbaut and Peaucelle (2006b) proposed a new criterion for testing the stability of a linear time delay system, where the linear time-delay system with zero delay \((h=0)\) is stable. This article is based on the quadratic separation framework (Peaucelle, Henrion, and Arzelier 2005) and discretisation of the delay.

There have been many studies about linear time-delay systems with a single delay (Brierley et al. 1982; Gouaisbaut and Peaucelle 2006a, 2006b; Gu 1997; Han and Gu 2001; Han et al. 2003; Li and Souza 1997; Moon et al. 2001; Mori et al. 1981; Mori, Fukuma, and Kuwahara 1982; Mori and Kokame 1989; Niculescu et al. 1995, 1999; Park 1999; Su et al. 1994). However, linear time-delay systems with multiple delays have not been much studied (Fridman and Shaked 2002, 2003; Gu 1997, Kolmanovskii and Richard 1999; Lewis and Anderson 1980; Sipahi and Olgac 2005; Vyhlidal and Zitek 2006; Suh and

Time-delay systems can be divided into two types, retarded or neutral system. The retarded system contains delays only in its states, whereas the neutral system contains delays both in its states and in the derivatives of its states, so-called neutral delays. During the past several years, some researchers presented stability criteria for linear neutral time-delay systems with a single delay (Fridman 2006; He et al. 2004; Han 2005; Wu, He, She, and Liu 2004) and multiple delays (Fridman 2001; Fridman and Shaked 2003). Almost all stability criteria are delay-dependent with respect to state delays and delay-independent with respect to neutral delays. The delay-dependent stability criterion with respect to state delays and neutral delays was proposed by He et al. (2004). Fridman (2006) and Han (2005) proposed the stability criteria for linear neutral systems with a single delay based on the Gu’s discretised Lyapunov functional approach in Gu (1997).

In this article, delay-dependent stability criteria for linear retarded and neutral systems with multiple delays are proposed by employing the Lyapunov–Krasovskii functional approach and integral inequality. By the N-segmentation of delay length, we obtain more relaxed results on the delay bounds which guarantee the asymptotic stability of the linear retarded and neutral systems with multiple delays. The stability criterion is formulated in terms of LMIs (linear matrix inequalities). Simulation results show that the proposed stability criteria for retarded and neutral systems with multiple delays are less conservative than those of existing results in Fridman and Shaked (2002, 2003), and Kolmanovskii and Richard (1999).

2. Main results
2.1. Retarded systems

We start the analysis with the notationally easier case with two delays. The principles will be clear and will be extended to the fully general multiple delay case in the later parts. Consider the linear retarded systems with multiple delays described by the following equation

\[
x(t) = Ax(t) + \sum_{i=1}^{\gamma} B_i x(t - h_i) \\
\]

where \( A \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times n}, x(t) \in \mathbb{R}^n \) for all \( t \in [0, \infty), h_i \in \mathbb{R^+} \) and \( h_{\text{max}} = \max_i h_i \).

In this article, we are interested in developing the sufficient conditions to guarantee the asymptotic stability of the system (1) for a certain range of time-delays. In order to introduce the main results, we need the following Lemma:

**Lemma 1:** For a positive definite and diagonal matrix \( D \), the following holds:

\[
\left( \int_a^b F_T(\alpha) d\alpha \right) R \left( \int_a^b F(\alpha) d\alpha \right) \leq (b - a) \left( \int_a^b F_T(\alpha) R F(\alpha) d\alpha \right)
\]

with

\[
F(\alpha) = \begin{bmatrix} f_1(\alpha) \\ \vdots \\ f_n(\alpha) \end{bmatrix} \quad \alpha \in \mathbb{R}^n \quad \text{where } f_i(\alpha) \text{ is a real and integrable function on } [a, b]
\]

\( F_T(\alpha) \): transpose of \( F(\alpha) \)

\( R = \text{diag}(r_1, \ldots, r_n) \) where \( r_i \) is a real constant for \( i = 1, \ldots, n \)

**Proof of Lemma 1:** As it follows from the Cauchy–Schwartz inequality, we show

\[
\left( \int_a^b f(\alpha) g(\alpha) d\alpha \right) \leq \left( \int_a^b f^2(\alpha) d\alpha \right) \left( \int_a^b g^2(\alpha) d\alpha \right)
\]

where \( f(\alpha) \) and \( g(\alpha) \) are \( L_2 \) on \([a, b]\). Let \( f(\alpha) = f_i(\alpha) \) and \( g(\alpha) = 1 \), then (3) becomes

\[
\left( \int_a^b f_i(\alpha) d\alpha \right)^2 < (b - a) \left( \int_a^b f_i^2(\alpha) d\alpha \right)
\]
Multiplying a real constant, \( r_i \), on both side of (4) for \( i = 1, \ldots, n \), then

\[
  r_i \left( \int_a^b f_i(\alpha) \, d\alpha \right) \left( \int_a^b f_i(\alpha) \, d\alpha \right) < r_i(b-a) \left( \int_a^b f_i(\alpha)f_i(\alpha) \, d\alpha \right)
\]

From the summation for both side of Equation (5), respectively, one can get

\[
  \sum_{i=1}^n \left\{ \left( \int_a^b f_i(\alpha) \, d\alpha \right) r_i \left( \int_a^b f_i(\alpha) \, d\alpha \right) \right\} < \sum_{i=1}^n \left( b-a \right) \left( \int_a^b f_i(\alpha)r_i f_i(\alpha) \, d\alpha \right) \tag{6}
\]

Note that (6) can be expressed as

\[
  \left( \int_a^b F^T(\alpha) \, d\alpha \right) R \left( \int_a^b F(\alpha) \, d\alpha \right) < (b-a) \left( \int_a^b F^T(\alpha)R F(\alpha) \, d\alpha \right) \tag{7}
\]

This completes the proof. \( \square \)

Based on the above Lemma 1, we obtain the following result.

**Lemma 2:** Consider the linear time-delay system with a single delay described by the following equation

\[
  \dot{x}(t) = Ax(t) + Bx(t-h) \quad \text{for all } t \in [-h, 0] \quad \text{(8)}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}, x(t) \in \mathbb{R}^n \) for all \( t \in [0, \infty) \) and \( h \in \mathbb{R}^+ \).

If there exist \( P \), \( Q \), and \( R \) in \( \mathbb{R}^{n \times n} \) where \( P = P^T > 0, Q = Q^T > 0, \) and \( R = R^T > 0 \) with \( R = \text{diag}(r_1, \ldots, r_n) \) which satisfy the LMI condition

\[
  K = \begin{bmatrix} A^T P + PA + Q & hA^T RB + PB + R/h \\ hA^T RA - R/h & hB^T RA + B^T P + R/h \end{bmatrix} < 0, \tag{9}
\]

then the system (8) is asymptotically stable.

**Proof of Lemma 2:** To extend the Lyapunov stability theorem to the time delay systems, the Lyapunov–Krasovskii functional is constructed as

\[
  V = V_1 + V_2 + V_3
  = \dot{x}^T(t)Px(t) + \int_t^\infty \dot{x}^T(\alpha)Qx(\alpha) \, d\alpha + \int_{t-h}^0 \int_t^{t-h} \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \, d\beta \tag{10}
\]

where \( V_1 \) is a simple quadratic form, \( V_2 \) is introduced to determine the delay-independent stability, and \( V_3 \) is added to determine the delay-dependent stability (Kolmanovskii and Richard 1999; Park 1999; Moon et al. 2001; Fridman and Shaked 2003; and Wu et al. 2004). The time-derivatives of \( V_1, V_2, \) and \( V_3 \) in Equation (10) are

\[
  \dot{V}_1 = \dot{x}^T(t)Px(t) + \dot{x}^T(t)R\dot{x}(t)
  = \dot{x}^T(t)(A^T P + PA)x(t) + \dot{x}^T(t)P Bx(t-h) + \dot{x}^T(t-h)B^T Px(t)
\]

\[
  \dot{V}_2 = \dot{x}^T(t)Qx(t) - \dot{x}^T(t-h)Qx(t-h)
\]

\[
  \dot{V}_3 = \frac{d}{dt} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\} \tag{11}
\]

Using Lemma 1, we show the following inequality:

\[
  \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\} < \frac{h}{1} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\}
\]

Then, we have

\[
  \frac{1}{h} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\} < \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\}
\]

and

\[
  \dot{V}_3 = h\ddot{x}^T(t)R\ddot{x}(t) - \int_{t-h}^t \dot{x}(\alpha)R\dot{x}(\alpha) \, d\alpha \tag{12}
\]

\[
  \leq h \left\{ \int_{t-h}^t \dot{x}^T(\alpha)A^T RAx(\alpha) + \dot{x}^T(\alpha)A^T RBx(\alpha) \right\} + \dot{x}^T(t-h)B^T RX(t-h)
\]

\[
  + \dot{x}^T(t-h)B^T RX(t-h)
\]

\[
  \leq -\frac{1}{h} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\} - \dot{x}^T(t-h)B^T RX(t-h)
\]

\[
  - \dot{x}^T(t-h)B^T RX(t-h)
\]

\[
  \leq -\frac{1}{h} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\}
\]

\[
  \leq -\frac{1}{h} \left\{ \int_{t-h}^t \dot{x}^T(\alpha)R\dot{x}(\alpha) \, d\alpha \right\}
\]
Therefore, the time-derivative of Equation (10) is shown as

\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3
\]

\[
\leq z^T \left[ \begin{array}{cccc}
A^T P + PA + Q + & hA^T RB + PB + R/h \\
hA^T RA - R/h & hB^T RA + B^T P + R/h - hB^T RB - Q - R/h
\end{array} \right] z
\]

\[
= z^T K z
\]

where \( z = [x^T(t) \ x^T(t-h)]^T \). Inequality (9) immediately implies the asymptotic stability of the system (8). \( \Box \)

Now, we extend the result of Lemma 2 by introducing an N-segmentation method to reduce the conservatism of the LMI condition. Segmenting the delay length \( h \) into \( N \) pieces of the same length improves the knowledge on the system and allows us to build a new Lyapunov-Krasovskii functional. Hence, the previous LMI condition in (9) is expanded into the following LMI condition where \( n(N+1) \times n(N+1) \) matrix is used.

**Theorem 1:** If there exist \( P, Q_i, \) and \( R_i \in \mathbb{R}^{n \times n} \) for \( i = 1, \ldots, N \) where \( P = P^T > 0, \) \( Q_i = Q_i^T > 0, \) and \( R_i = R_i^T > 0 \) with \( R_i = \text{diag}(r_{i1}, \ldots, r_{ih}) \) which satisfy the LMI condition for a positive integer \( N > 1 \)

\[
\begin{align*}
W_{00} & = A^T P + PA + Q + \sum_{i=1}^{N} dA^T R_i A - R_i/d, \\
W_{0N} & = PB + \sum_{i=1}^{N} dA^T R_i B, \\
W_{pp} & = Q_{p+1} - Q_p - R_{p+1}/d - R_p/d, \\
W_{NN} & = \sum_{i=1}^{N} dB^T R_i B - Q_N - R_N/d.
\end{align*}
\]

then the system (8) is asymptotically stable.

**Proof of Theorem 1:** The Lyapunov–Krasovskii functional is constructed as

\[
V = V_1 + V_2 + V_3
\]

\[
= x^T(t)P x(t) + \sum_{i=1}^{N} \int_{t-a}^{t-b} x^T(\alpha) Q_{N+1-i} x(\alpha) d\alpha
\]

\[
+ \sum_{i=1}^{N} \int_{t-b}^{t} \dot{x}^T(\alpha) R_{N+1-i} \dot{x}(\alpha) d\beta
\]

where \( a = h_{N+1-i} = d \times (N+1-i), \) \( b = h_{N-i} = d \times (N-i), \) \( d = h/N, i = 1, \ldots, N, h_0 = 0, \) and \( h_N = h. \)

Equation (12) can also be derived from Equation (10). Let the time delay interval \([-h, 0]\) be divided into \( N \) segments \([-h_{i+1}, h_{i}]\) of the equal length \( d \) in Equation (10) and let positive definite matrices, \( Q_i \) and \( R_i \) be defined for each \( i \)-th segment, where \( i = 1, \ldots, N. \) Then, the number of positive definite matrices and state vectors become \( 2N+1 \) and \( N+1, \) respectively. Since there is a similarity between Equations (10) and (12), the time-derivative of the Lyapunov–Krasovskii functional (12) can be easily derived based on the same method used in Lemma 2.

The time-derivatives of \( V_1, V_2, \) and \( V_3 \) in Equation (12) are

\[
\begin{align*}
\dot{V}_1 & = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) \\
& = x^T(t) [A^T P + PA] x(t) + x^T(t) P B x(t-h) \\
& \quad + x^T(t-h) B^T P x(t)
\end{align*}
\]

\[
\dot{V}_2 = \sum_{i=1}^{N} \{ x^T(t-h_{N-i}) Q_{N+1-i} x(t-h_{N-i}) - x^T(t-h_{N+1-i}) Q_{N+1-i} x(t-h_{N+1-i}) \}
\]

\[
\dot{V}_3 = \sum_{i=1}^{N} \int_{t-a}^{t-b} \dot{x}^T(\alpha) R_{N+1-i} \dot{x}(\alpha) d\beta
\]

\[
= \sum_{i=1}^{N} \left\{ d \dot{x}^T(t) R_{N+1-i} \dot{x}(t) - \int_{t-a}^{t-b} \dot{x}^T(\alpha) R_{N+1-i} \dot{x}(\alpha) d\alpha \right\}
\]

Using Lemma 1, we show the following inequality:

\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3
\]

\[
\leq x^T(t) [A^T P + PA] x(t) + x^T(t) P B x(t-h) \\
+ x^T(t-h) B^T P x(t)
\]
\[ + \sum_{i=1}^{N} \left\{ x^T(t-h_{N-i})Q_{N+1-i}x(t-h_{N-i}) 
abla \right. \\
abla \left. x^T(t-h_{N-i})Q_{N+1-i}x(t-h_{N-i}) \right\} \\
abla + \sum_{i=1}^{N} d(t)A^TR_{N+1,i}Ax(t) \\
abla + x^T(t)A^TR_{N+1-i}Bx(t-h) + x^T(t-h)B^TR_{N+1-i}Ax(t) \\
abla + x^T(t-h)B^TR_{N+1-i}Bx(t-h) \right] \\
abla - \sum_{i=1}^{N} \left\{ \frac{1}{d} x^T(t-h_{N-i})R_{N+1-i}x(t-h_{N-i}) 
abla \right. \\
abla \left. - \frac{1}{d} x^T(t-h_{N-i})R_{N+1-i}x(t-h_{N-i}) \right\} \\
abla - \frac{1}{d} x^T(t-h_{N-i})R_{N+1-i}x(t-h_{N-i}) \\
abla + \frac{1}{d} x^T(t-h_{N-i})R_{N+1-i}x(t-h_{N-i}) \right\} \\
abla = z^TKSz \]

where \( z = \left[ x^T(t), x^T(t-h_1), \ldots, x^T(t-h_N) \right]^T \), \( x(t-h) \in \mathbb{R}^n \), \( z \in \mathbb{R}^{n(N+1)} \), and \( K_S \in \mathbb{R}^{n(N+1) \times n(N+1)} \). Hence, inequality (11) immediately implies the asymptotic stability of the system (8). \( \square \)

**Remark 1:** The LMI condition in Lemma 2 is the same as the LMI condition in Theorem 1 at \( N = 1 \).

**Remark 2:** The dimension of the LMI condition and the number of decision variables in (11) are \( n(N+1) \times n(N+1) \) and \( (N+1) \left( \sum_{i=1}^{n} \hat{t} \right) + nN \). The dimension of the LMI condition and the number of decision variables in Gouaisbaut and Peaucelle (2006a) are \( 2nN \times 2nN \) and \( (2N+1)nN(nN+1)/2 \), which are about two times and \( N \) times of the LMI condition (11), respectively. The dimension of the LMI condition in Gouaisbaut and Peaucelle (2006b) is the same as the dimension of the LMI condition (11), but the number of decision variables in Gouaisbaut and Peaucelle (2006b) is \( nN(nN+1)/2 + n(n+1) \) which is larger than the number of decision variables in (11). The dimension of the LMI condition is smaller in our result and there are a lot more decision variables in Gouaisbaut and Peaucelle (2006a, 2006b) compared with ours. Thus, we can say that, computational-wise, (11) is more effective than the LMI conditions in Gouaisbaut and Peaucelle (2006a, 2006b).

Now, we consider the two-delay system in (1) and apply our \( N \)-segmentation method and integral inequality to obtain the following LMI condition.

**Theorem 2:** If there exist \( P, Q_i, R_k \in \mathbb{R}^{n \times n} \) for \( i = 1, 2 \) and \( k = 1, \ldots, N_i \) where \( P = P^T > 0, Q_i^T = (Q_i^T)^T > 0, \) and \( R_k^T = (R_k^T)^T > 0 \) with \( R_k^T = \text{diag} \times \{(r_k^1), \ldots, (r_k^k)\} \) which satisfy the LMI condition for positive integers \( N_i > 1 \)

\[ K_M = \begin{bmatrix} K_{00} & K_{01} & K_{02} \\
K_{01}^T & K_{11} & K_{12} \\
K_{02}^T & K_{12}^T & K_{22} \end{bmatrix} < 0 \] (13)

\[ \begin{align*}
K_{00} &= A^TP + PA + Q_1^T + Q_2^T \\
&+ 2 \sum_{m=1}^{N_m} \sum_{k=1}^{nN_m} d_m A^T R_m^k A - 2 \sum_{m=1}^{N_m} R_m^k/d_m, \\
K_{0i} &= \begin{bmatrix} R_i^1/d_i & 0 & \cdots & 0 \\
W_i(1,1) & R_i^2/d_i & \cdots & 0 \\
& & \ddots & \vdots \\
& & & R_i^{N_i}/d_i \end{bmatrix}, \\
K_{ii} &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & W_i(N_i, N_i) \\
0 & \cdots & 0 & \sum_{m=1}^{N_m} \sum_{k=1}^{nN_m} d_m B_m^T R_m^k B_m \end{bmatrix}, \\
W(p, p) &= Q_i^{p+1} - Q_i^p - R_i^{p+1}/d_i - R_i^p/d_i, \\
p &= 1, \ldots, (N_i - 1), \\
W_i(N_i, N_i) &= \sum_{m=1}^{N_m} \sum_{k=1}^{nN_m} d_m B_m^T R_m^k B_m - Q_i^{N_i} - R_i^{N_i}/d_i, \\
\end{align*} \]

then the system (1) is asymptotically stable.

**Proof of Theorem 2:** The Lyapunov–Krasovskii functional is constructed as

\[ V = V_1 + V_2 + V_3 \]

\[ = x^T(t)Px(t) + \sum_{i=1}^{N_i} \sum_{k=1}^{n} \int_{t-a_i}^{t-b_i} x^T(\alpha)Q_i^{N_i+1-k}x(\alpha) \, d\alpha \]

\[ + \sum_{i=1}^{N_i} \sum_{k=1}^{n} \int_{t-a_i}^{t-b_i} \int_{t+\beta}^{t+b_i} x^T(\alpha)R_i^{N_i+1-k}z(\alpha) \, d\alpha \, d\beta \] (14)

where \( a_i = h_i^{N_i+1-k} = d_i \times (N_i + 1 - k), \) \( b_i = h_i^{N_i-k} = d_i \times (N_i - k), \) \( d_i = h_i/N_i, \) \( h_i^0 = 0, \) and \( h_i^{N_i} = h_i \)
for \( i = 1, 2 \) and \( k = 1, \ldots, N_i \). As one can see in Equation (14), the numbers of positive definite matrices and state vectors become \( 1 + 2 \sum_{i=1}^{N_i} N_i \) and \( 1 + \sum_{i=1}^{N_i} N_i \), respectively. The time-derivative of Equation (14) is shown as

\[
\dot{V}_1 = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t)
\]

\[
= \left\{ Ax(t) + \sum_{i=1}^{2} B_i x(t - h_i) \right\}^T P x(t)
\]

\[
+ x^T(t)P \left\{ Ax(t) + \sum_{i=1}^{2} B_i x(t - h_i) \right\}
\]

\[
= x^T(t) \left\{ A^2 P + PA \right\} x(t)
\]

\[
+ \sum_{i=1}^{2} \sum_{k=1}^{N_i} \left\{ x^T(t - h_i) B_i^T P x(t) + x^T(t) P B_i x(t - h_i) \right\}
\]

\[
+ \sum_{i=1}^{2} \sum_{k=1}^{N_i} \left\{ x^T \left( t - h_i^{N_i+1-k} \right) Q_i^{N_i+1-k} x(t - h_i^{N_i+1-k}) \right\}
\]

\[
- x^T \left( t - h_i^{N_i+1-k} \right) Q_i^{N_i+1-k} x(t - h_i^{N_i+1-k})
\]

\[
+ \sum_{i=1}^{2} \sum_{k=1}^{N_i} \left\{ d_i \left[ Ax(t) + \sum_{m=1}^{2} B_m x(t - h_m) \right]^T R_i^{N_i+1-k} \right\}
\]

\[
\times \left[ Ax(t) + \sum_{m=1}^{2} B_m x(t - h_m) \right]
\]

\[
- \sum_{i=1}^{2} \sum_{k=1}^{N_i} \left\{ \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k}) \right\}
\]

\[
- \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k})
\]

\[
+ \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k})
\]

\[
= z^T K_M z
\]

where \( z = [x^T(t) \ x^T(t - h_1) \ \ldots \ x^T(t - h_{N_1})]^T \), \( x(t) \in \mathbb{R}^n \), \( z \in \mathbb{R}^{(1+\sum_{i=1}^{N_i})} \), and \( K_M \in \mathbb{R}^{(1+\sum_{i=1}^{N_i}) \times (1+\sum_{i=1}^{N_i})} \).

Hence, inequality (13) immediately implies the asymptotic stability of the system (1).

**Remark 3:** Based on Theorem 2, we can obtain the delay-dependent stability criterion for a system with \( l \) multiple delays, where \( l \) is a positive integer. Consider the following linear time-delay system with \( l \) multiple delay

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{l} B_i x(t - h_i)
\]

(15)

\[
\forall t \in [-h_{\text{max}}, 0]
\]

where \( A \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times n} \), \( x(t) \in \mathbb{R}^n \) for all \( t \in [0, \infty] \), \( h_i \in \mathbb{R}^+ \) and \( h_{\text{max}} = \max h_i \).

The Lyapunov-Krásovskii functional is constructed as

\[
V = V_1 + V_2 + V_3
\]

\[
= x^T(t)Px(t) + \sum_{i=1}^{l} \int_{t-h_i}^{t} x^T(\alpha) Q_i^{N_i+1-k} x(\alpha) d\alpha
\]

\[
+ \sum_{i=1}^{l} \sum_{k=1}^{N_i} \left\{ d_i \left[ Ax(t) + \sum_{m=1}^{2} B_m x(t - h_m) \right]^T R_i^{N_i+1-k} \right\}
\]

\[
\times \left[ Ax(t) + \sum_{m=1}^{2} B_m x(t - h_m) \right]
\]

\[
- \sum_{i=1}^{l} \sum_{k=1}^{N_i} \left\{ \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k}) \right\}
\]

\[
- \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k})
\]

\[
+ \frac{1}{d_i} x^T(t - h_i^{N_i+1-k}) R_i^{N_i+1-k} x(t - h_i^{N_i+1-k})
\]

\[
= z^T K_M z
\]

where \( z = [x^T(t) \ x^T(t - h_1) \ \ldots \ x^T(t - h_{N_1})]^T \), \( x(t) \in \mathbb{R}^n \), \( z \in \mathbb{R}^{(1+\sum_{i=1}^{N_i})} \), and \( K_M \in \mathbb{R}^{(1+\sum_{i=1}^{N_i}) \times (1+\sum_{i=1}^{N_i})} \).
which satisfy the LMI condition for positive integers $N_i > 1$

\[
K_M = \begin{bmatrix}
K_{00} & \cdots & K_{0l} \\
\vdots & \ddots & \vdots \\
K_{0l} & \cdots & K_{ll}
\end{bmatrix} < 0 \tag{16}
\]

where

\[
d_i = h_i/N_i, \quad i = 1, \ldots, l,
\]

\[
K_{00} = A^T P + PA + \sum_{m=1}^{l} Q_m^1 + \sum_{m=1}^{l} \sum_{k=1}^{N_m} d_m A^T R_{m}^k A - \sum_{m=1}^{l} R_{m}^1/d_m,
\]

\[
K_{0l} = \begin{bmatrix}
R_1^l/d_i & 0 & \cdots & 0 & PB_l + \sum_{m=1}^{l} \sum_{k=1}^{N_m} d_m A^T R_{m}^k B_l \\
W_l(1, 1) & R_2^l/d_i & 0 & \cdots & 0 \\
& R_2^l/d_i & \ddots & \vdots & \vdots \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
0 & \cdots & 0 & R_1^l/d_i & W_l(N_l; N_l)
\end{bmatrix},
\]

\[
K_{ll} = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \sum_{m=1}^{l} \sum_{k=1}^{N_m} d_m B_l^T R_{m}^k B_l \\
0 & \cdots & 0 & \sum_{m=1}^{l} \sum_{k=1}^{N_m} d_m B_l^T R_{m}^k B_l - Q_1^N - R_Y^1/d_i
\end{bmatrix},
\]

\[
q = (i + 1), \ldots, l,
\]

\[
W_l(p, p) = Q_l^p + Q_l^N - R_l^1/d_i - R_l^p/d_i,
\]

\[
p = 1, \ldots, (N_l - 1),
\]

\[
W_l(N_l; N_l) = \sum_{m=1}^{l} \sum_{k=1}^{N_m} d_m B_l^T R_{m}^k B_l - Q_1^N - R_Y^1/d_i
\]

with $K_{00} \in \mathbb{R}^{n \times n}$, $K_{0l} \in \mathbb{R}^{n \times (N - N_l)}$, $K_{ll} \in \mathbb{R}^{(n - N_l) \times (n - N_l)}$, and $K_M \in \mathbb{R}^{(1 + \sum_{m=1}^{l} N_m) \times n \times (1 + \sum_{m=1}^{l} N_m)}$, then the system (15) is asymptotically stable.

### 2.2. Neutral systems

Consider the linear neutral systems with multiple delays described by the following equation

\[
\begin{align*}
\dot{x}(t) - \sum_{j=1}^{l} C_j \dot{x}(t - \tau_j) &= A x(t) + \sum_{j=1}^{l} B_j x(t - h_j) \\
x(t) &= \phi(t), \quad \forall t \in [-h_{\max}, 0]
\end{align*} \tag{17}
\]

where $A \in \mathbb{R}^{n \times n}$, $B_j \in \mathbb{R}^{m \times n}$, $C_j \in \mathbb{R}^m \times n$, $x(t) \in \mathbb{R}^n$ for all $t \in [0, \infty]$, $h_i \in \mathbb{R}^+$, $\tau_j \in \mathbb{R}^+$, and $h_{\max} = \max_i (h_i, \tau_j)$. In order to develop a delay-dependent stability criterion with respect to state delays, $h_i$, and neutral delays, $\tau_j$, for the system (17), we construct the Lyapunov–Krasovskii functional using the N-segmentation method as follows:

\[
V = V_1 + V_2 + V_3 + V_4 + V_5 + V_6 \tag{18}
\]

where

\[
V_1 = x^T(t) P x(t), \quad V_2 = \sum_{i=1}^{l} \sum_{k=1}^{N_i} \int_{-h_i}^{0} x^T(\alpha) Q_i^{N_i+1-k} x(\alpha) d\alpha
\]

\[
V_3 = \sum_{i=1}^{l} \sum_{k=1}^{N_i} \int_{-\tau_i}^{0} \int_{t-\tau_i}^{t} x^T(\alpha) R_i^{N_i+1-k} x(\alpha) d\alpha d\beta
\]

\[
V_4 = \sum_{j=1}^{l} \sum_{k=1}^{N_j} \int_{-\gamma_j}^{0} \int_{t-\gamma_j}^{t} x^T(\alpha) S_j^{N_j+1-k} x(\alpha) d\alpha d\beta
\]

\[
V_5 = \sum_{j=1}^{l} \sum_{k=1}^{N_j} \int_{-\gamma_j}^{0} \int_{t-\gamma_j}^{t} x^T(\alpha) T_j^{N_j+1-k} x(\alpha) d\alpha d\beta
\]

\[
V_6 = \sum_{j=1}^{l} \sum_{k=1}^{N_j} \int_{-\gamma_j}^{0} \int_{t-\gamma_j}^{t} x^T(\alpha) M_j^{N_j+1-k} x(\alpha) d\alpha d\beta
\]

with $d_i = h_i/N_i$, $g_j = \tau_j/N_j$, $a_i = h_i^{N_i+1-k}$, $b_i = h_i^{N_i+1-k}$, $h_i = h_i^{N_i+1-k}$, $h_i^0 = 0$, $h_i^N = h_i$, $r_j = \tau_j^{N_j+1-k}$, $g_j = (N_i + 1 - k)$, $\lambda_j = \tau_j^{N_j+1-k}$, $\lambda_j = \tau_j^{N_j+1-k}$, $\tau_j^0 = 0$, and $\tau_j^N = \tau_j$ for $i = 1, \ldots, l$, $j = 1, \ldots, e$, and $k = 1, \ldots, N_i$. The new terms $V_4$, $V_5$, and $V_6$, comparatively to (16), are due to the neutral delays. The following result is based on Theorem 2 and Remark 3.

**Theorem 3:** If there exist $P$, $Q_i^p$, $R_i^p$, $S_i^k$, $T_i^k$, and $M_i^k \in \mathbb{R}^{n \times n}$ for $i = 1, \ldots, l$, $j = 1, \ldots, e$ and $k = 1, \ldots, N_i$ where $P = P^T > 0$, $Q_i^p = (Q_i^p)^T > 0$, $R_i^p = (R_i^p)^T > 0$, $S_i^k = (S_i^k)^T > 0$, $T_i^k = (T_i^k)^T > 0$ and $M_i^k = (M_i^k)^T > 0$ with $R_i^p = \text{diag}((r_i^p)^T, \ldots, (r_i^p)^T)$ and $T_i^k = \text{diag}((t_i^k)^T, \ldots, (t_i^k)^T)$ which satisfy the LMI condition for a positive integer $N > 1$

\[
K_N = \begin{bmatrix}
K_{00} & K_0 & H_0 & L_0 \\
K_0^T & K_1 & 0 & L_H \\
H_0^T & 0 & H_1 & 0 \\
L_H^T & L_H^T & 0 & L_1
\end{bmatrix} < 0 \tag{19}
\]

with $d_i = h_i/N_i$, $g_j = \tau_j/N_j$, $i = 1, \ldots, l$, $j = 1, \ldots, e$,

\[
K_{00} = A^T P + PA + \sum_{m=1}^{l} Q_m^1 - \sum_{m=1}^{l} R_{m}^1/d_m + \Phi_1,
\]
\[
\begin{align*}
\Phi_1 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m A^T R_{mk}^k A + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m A^T T_{mk}^k A \\
&+ \sum_{m=1}^{e} A^T S_{m}^1 A, \\
K_0 &= \begin{bmatrix} K_{01} & \cdots & K_{0e} \end{bmatrix}, \\
K_{0i} &= \begin{bmatrix} R_i^1/d_i & 0 & \cdots & 0 \ PB_i + \Phi_2 \end{bmatrix}, \\
\Phi_2 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m A^T R_{mk}^k B_i + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m A^T T_{mk}^k B_i \\
&+ \sum_{m=1}^{e} A^T S_{m}^1 B_i, \\
H_0 &= \begin{bmatrix} H_{01} & \cdots & H_{0e} \end{bmatrix}, \quad H_{0j} = \begin{bmatrix} T_j^1/g_j & 0 & \cdots & 0 \end{bmatrix}, \\
L_0 &= \begin{bmatrix} L_{01} & \cdots & L_{0e} \end{bmatrix}, \quad L_{0j} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}, \\
\Phi_3 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m A^T R_{mk}^k C_j + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m A^T T_{mk}^k C_j \\
&+ \sum_{m=1}^{e} A^T S_{m}^1 C_j, \\
K_i &= \begin{bmatrix} K_{i1} & \cdots & K_{i,l} \\
\vdots & \ddots & \vdots \\
K_{il}^T & \cdots & K_{il} \end{bmatrix}, \\
K_{ij} &= \begin{bmatrix} W_j(1, 1) & R_j^1/d_j & 0 & \cdots & 0 \\
R_j^2/d_j & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & R_j^l/d_j & W_j(N, N) \end{bmatrix}, \\
W_j(p, p) &= Q_j^{p+1} - Q_j^p - R_j^{p+1}/d_j - R_j^p/d_j, \\
p = 1, \ldots, (N - 1), \\
W_j(N, N) &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m B_i^T R_{mk}^k B_i + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m B_i^T T_{mk}^k B_i \\
&+ \sum_{m=1}^{e} B_i^T S_{m}^1 B_i - Q_j^N - R_j^N/d_j, \\
K_{iq} &= \begin{bmatrix} 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \Phi_4 \end{bmatrix}, \quad q = (i + 1), \ldots, l, \\
\Phi_4 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m B_i^T R_{mk}^k B_q + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m B_i^T T_{mk}^k B_q \\
&+ \sum_{m=1}^{e} B_i^T S_{m}^1 B_q, \\
H_1 &= \text{diag}(H_{1j}, \ldots, H_{e,j}), \\
H_{ij} &= \begin{bmatrix} Y_j(1, 1) & T_j^1/g_j & 0 & \cdots & 0 \\
T_j^2/g_j & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & T_j^N/g_j & Y_j(N, N) \\
0 & \cdots & 0 & T_j^N/g_j, \quad Y_j(p, p) = M_j^{p+1} - M_j^p - T_j^{p+1}/g_j - T_j^p/g_j, \\
p = 1, \ldots, (N - 1), \\
Y_j(N, N) &= -M_j^N - T_j^N/g_j, \\
L_H &= \begin{bmatrix} L_H(1, 1) & \cdots & L_H(1, e) \\
\vdots & \ddots & \vdots \\
L_H(l, 1) & \cdots & L_H(l, e) \end{bmatrix}, \\
L_{H(i, j)} &= \begin{bmatrix} 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 0 \end{bmatrix}, \\
\Phi_5 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m B_i^T R_{mk}^k C_j + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m B_i^T T_{mk}^k C_j \\
&+ \sum_{m=1}^{e} B_i^T S_{m}^1 C_j, \\
L_1 &= \begin{bmatrix} L_{11} & \cdots & L_{1e} \\
\vdots & \ddots & \vdots \\
L_{le} & \cdots & L_{ee} \end{bmatrix}, \\
L_{ij} &= \text{diag}\{(S_j^2 - S_j^1), \ldots, (S_j^N - S_j^{N-1}), (\Phi_6 - S_j^N)\}, \\
\Phi_6 &= \sum_{m=1}^{l} \sum_{k=1}^{N} d_m C_j^T R_{mk}^k C_j + \sum_{m=1}^{e} \sum_{k=1}^{N} g_m C_j^T T_{mk}^k C_j \\
&+ \sum_{m=1}^{e} C_j^T S_{m}^1 C_j.
\end{align*}
\]
\begin{equation*}
L_{jq} = \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0
\end{bmatrix}, \quad q = (j + 1), \ldots, e,
\end{equation*}

\begin{equation*}
\Phi_7 = \sum_{m=1}^{l} \sum_{k=1}^{N} d_m C_j^T R_m^k C_q + \sum_{m=1}^{e} \sum_{k=1}^{N} B_m C_j^T T_m^k C_q
+ \sum_{m=1}^{e} C_j^T S_m^k C_q,
\end{equation*}

then the system (17) is asymptotically stable.

**Proof of Theorem 3:** The time-derivative of Equation (18) is shown as

\[
\dot{V}_1 = x^T(t) \left[ A^T P + P A \right] x(t)
+ \sum_{i=1}^{l} \left\{ x^T(t - h_i) B_i^T P x(t) + x^T(t) P B_i x(t - h_i) \right\}
+ \sum_{j=1}^{e} \left\{ \dot{x}^T(t - \tau_j) C_j^T P x(t) + x^T(t) P C_j \dot{x}(t - \tau_j) \right\}
\]

\[
\dot{V}_2 = \sum_{i=1}^{l} \sum_{k=1}^{N} \left\{ x^T(t - h_i^{N-k}) Q_i^{N+1-k} x(t - h_i^{N-k})
- x^T(t - h_i^{N-k}) Q_i^{N+1-k} x(t - h_i^{N+1-k}) \right\}
\]

\[
\dot{V}_3 = \sum_{i=1}^{l} \sum_{k=1}^{N} \left\{ d_i \dot{x}^T(t) R_i^{N+1-k} \dot{x}(t) \right\}
- \sum_{i=1}^{l} \sum_{k=1}^{N} \left\{ \int_{t-h_i}^{t} \dot{x}^T(\alpha) R_i^{N+1-k} \dot{x}(\alpha) \, d\alpha \right\}
\]

\[
\leq \sum_{i=1}^{l} \sum_{k=1}^{N} \left\{ d_i \left[ A x(t) + \sum_{m=1}^{l} B_m x(t - h_m) \right]
+ \sum_{p=1}^{e} C_p \dot{x}(t - \tau_p) \right\}^T R_i^{N+1-k} \left[ A x(t)
+ \sum_{m=1}^{l} B_m x(t - h_m) \right]
+ \sum_{m=1}^{e} C_p \dot{x}(t - \tau_p) \right\}
\]

\[
\dot{V}_4 = \sum_{j=1}^{e} \sum_{k=1}^{N} \left\{ \dot{x}^T(t - \tau_j^{N-k}) S_j^{N+1-k} \dot{x}(t - \tau_j^{N-k})
- \dot{x}^T(t - \tau_j^{N+1-k}) S_j^{N+1-k} \dot{x}(t - \tau_j^{N+1-k}) \right\}
\]

\[
\dot{V}_5 = \sum_{j=1}^{e} \sum_{k=1}^{N} \left\{ g_j \dot{x}^T(t) T_j^{N+1-k} \dot{x}(t) \right\}
- \sum_{j=1}^{e} \sum_{k=1}^{N} \left\{ \int_{t-\gamma_j}^{t} \dot{x}^T(\alpha) T_j^{N+1-k} \dot{x}(\alpha) \, d\alpha \right\}
\]

\[
\leq \sum_{j=1}^{e} \sum_{k=1}^{N} \left\{ g_j \left[ A x(t) + \sum_{m=1}^{l} B_m x(t - h_m) \right]
+ \sum_{p=1}^{e} C_p \dot{x}(t - \tau_p) \right\}^T T_j^{N+1-k} \left[ A x(t)
+ \sum_{m=1}^{l} B_m x(t - h_m) \right]
+ \sum_{m=1}^{e} C_p \dot{x}(t - \tau_p) \right\}
\]

\[
\dot{V}_6 = \sum_{j=1}^{e} \sum_{k=1}^{N} \left\{ x^T(t - \tau_j^{N-k}) M_j^{N+1-k} x(t - \tau_j^{N-k})
- x^T(t - \tau_j^{N+1-k}) M_j^{N+1-k} x(t - \tau_j^{N+1-k}) \right\}
\]

As it follows from Theorem 2 and Remark 3, we show the following inequality:

\[
\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5 + \dot{V}_6 \leq z^T K_N z
\]

where

\[
z = \begin{bmatrix} x^T(t) & x_0^T(t) & x_i^T(t) & \dot{x}_0^T(t) \end{bmatrix}^T,
\]

\[
x_0^T(t) = \begin{bmatrix} x^T(t - h_1^{1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x^T(t - h_l^{1}) \end{bmatrix},
\]

\[
\dot{x}_0^T(t) = \begin{bmatrix} \dot{x}^T(t - \tau_1^{1}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \dot{x}^T(t - \tau_e^{1}) \end{bmatrix},
\]

\[
\dot{x}_i^T(t) = \begin{bmatrix} \dot{x}^T(t - \tau_1^{i}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \dot{x}^T(t - \tau_e^{i}) \end{bmatrix},
\]

\[
\dot{x}_i^T(t) = \begin{bmatrix} \dot{x}^T(t - \tau_1^{i}) & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \dot{x}^T(t - \tau_e^{i}) \end{bmatrix},
\]

\[
z \in \mathbb{R}^{(1 + IN + 2eN)} 
\quad \text{and} 
K_N \in \mathbb{R}^{(1 + IN + 2eN) \times (1 + IN + 2eN)}. 
\]

Hence, inequality (19)
As you can see in Theorem 3, LMI directly implies the asymptotic stability of the system (17).

Remark 4: As you can see in Theorem 3, LMI condition (18) depends not only on state delays $h_i$ but also on neutral delays $\tau_j$. However, the stability criterion in Fridman and Shaked (2003) is independent of state delays $h_i$ and neutral delays $\tau_j$.

3. Numerical examples

3.1. Retarded systems

First, we compare our result with several existing results in a single delay case. As we mentioned earlier, Han et al. (2003) proposed an improved delay-dependent stability criterion for a time-delay system with single-delay.

Example 1: Consider the second order system with a single delay studied in Fridman (2001), Han et al. (2003), and Kolmanovskii and Richard (1999).

\[ \dot{x}(t) = \begin{bmatrix} -1.0 & 0.5 \\ -0.5 & -1.0 \end{bmatrix} x(t) + \begin{bmatrix} -2.0 & 0.0 \\ -2.0 & -2.0 \end{bmatrix} x(t - h) \]

We set several segmentation values as $N = 1, 2, 4, 10, 15, 20$ and observe that less conservative delay bound on $h$ is obtained as $N$ increases (Table 1). However, for $N \geq 4$, there is no big difference in the obtained bounds on $h$. Thus, we may recommend the segmentation value $N = 4$ to avoid the complexity of the LMI condition. As shown in Table 2, we obtain the same results as Han’s one. And also, Table 2 shows that our proposed method yields less conservative bound on $h$ than other existing results.

<table>
<thead>
<tr>
<th>Reference</th>
<th>Max. $h$</th>
</tr>
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<tbody>
<tr>
<td>Kolmanovskii and Richard (1999)</td>
<td>0.2716</td>
</tr>
<tr>
<td>Park (1999)</td>
<td>0.3434</td>
</tr>
<tr>
<td>Fridman and Shaked (2002)</td>
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<tr>
<td>Fridman and Shaked (2003)</td>
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<td>Wu et al. (2004)</td>
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<tr>
<td>He et al. (2004)</td>
<td>0.3449</td>
</tr>
<tr>
<td>Han et al. (2003)</td>
<td>0.3645</td>
</tr>
</tbody>
</table>

Next, we consider linear time-delay systems with two delays and compare our result with other existing results. Since Han’s criterion in Han et al. (2003) is applicable only for single-delay systems, Han’s criterion cannot be compared with ours for multiple delay systems, although Han’s criterion is as good as ours for single-delay systems.

Example 2: Consider the second order system with two delays

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} -1.0 & 0.5 \\ -0.5 & -1.0 \end{bmatrix} x(t) + \begin{bmatrix} -2.0 & 0.0 \\ -2.0 & -2.0 \end{bmatrix} x(t - h_1) \\
&\quad + \begin{bmatrix} 0.0 & 2.0 \\ 0.0 & -2.0 \end{bmatrix} x(t - h_2)
\end{align*}
\]

If $h_1 = h_2$, then one can get the stability boundary $h_1 = h_2 = 0.3685$ by MATLAB simulation. In order to get the stable region as a function of time delays, we construct the matrix $K_M$ in Theorem 2 for $N_1 = N_2 = 1$, $N_1 = N_2 = 2$, $N_1 = N_2 = 4$, and $N_1 = N_2 = 10$. Specifically, the matrix $K_M$ for $N_1 = N_2 = 4$ is constructed as the below Equation (20)

\[
K_M = \begin{bmatrix} K_{00} & K_{01} & K_{02} \\ K_{01}^T & K_{11} & K_{12} \\ K_{02}^T & K_{12}^T & K_{22} \end{bmatrix}
\]

where

\[
K_{00} = A^T P + PA + Q_1^1 + Q_2^1 \\
+ \sum_{i=1}^{4} \sum_{k=1}^{4} d_i A^T R_i^1 A - \sum_{i=1}^{4} R_i^1/d_i,
\]

\[
K_{01} = \begin{bmatrix} R_1^1/d_1 & 0 & 0 \\ 0 & PB_1 & \sum_{i=1}^{4} \sum_{k=1}^{4} d_i A^T R_i^1 B_1 \end{bmatrix},
\]

\[
K_{02} = \begin{bmatrix} R_2^1/d_2 & 0 & 0 \\ 0 & PB_2 & \sum_{i=1}^{4} \sum_{k=1}^{4} d_i A^T R_i^1 B_2 \end{bmatrix},
\]

\[
K_{11} = \begin{bmatrix} W_1(1, 1) & R_1^2/d_1 & 0 & 0 \\ R_2^2/d_1 & W_2(2, 2) & R_1^2/d_1 & 0 \\ 0 & R_1^2/d_1 & W_1(3, 3) & R_1^2/d_1 \\ 0 & 0 & R_1^2/d_1 & W_1(4, 4) \end{bmatrix},
\]

\[
K_{22} = \begin{bmatrix} W_2(1, 1) & R_2^2/d_2 & 0 & 0 \\ R_2^2/d_2 & W_2(2, 2) & R_2^2/d_2 & 0 \\ 0 & R_2^2/d_2 & W_2(3, 3) & R_2^2/d_2 \\ 0 & 0 & R_2^2/d_2 & W_2(4, 4) \end{bmatrix},
\]
\[ K_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{2} d_i R_i^1 R_i^2 B_2 \end{bmatrix} \]

with \( d_1 = h_1 / N_1, \quad d_2 = h_2 / N_2, \quad W_1(1, 1) = Q_1^1 - Q_1^2 - R_1^2 / d_1 - R_1^1 / d_1, \quad W_1(2, 2) = Q_2^1 - Q_2^2 - R_2^1 / d_1 - R_2^2 / d_1, \quad W_1(3, 3) = Q_1^1 - Q_1^2 - R_1^2 / d_1 - R_1^1 / d_1, \quad W_1(4, 4) = \sum_{i=1}^{2} \sum_{k=1}^{4} d_i R_i^1 R_i^2 B_1 - Q_1^1 - Q_2^1 - Q_2^2 / d_2 - R_2^1 / d_2, \quad W_2(3, 2) = Q_1^1 - Q_2^2 - R_2^1 / d_2 - R_2^2 / d_2, \quad W_2(2, 3) = Q_1^2 - Q_2^1 - R_2^1 / d_2 - R_2^2 / d_2 \), and \( W_2(4, 4) = \sum_{i=1}^{2} \sum_{k=1}^{4} d_i R_i^1 R_i^2 B_2 - Q_1^2 - R_2^2 / d_2 \).

Figure 1 depicts the stable region as a function of time delays for Example 2 by using MATLAB LMI Toolbox (Gahinet, Nemirovski, Laub, and Chilali 1995). From Figure 1(a), it can be seen that the stable region is expanded as \( N_1 \) and \( N_2 \) are increased. However, one can note that the rate of increment decreases as \( N_1 \) and \( N_2 \) are increased. The difference between \( N_1 = N_2 = 4 \) and \( N_1 = N_2 = 10 \) is quite small. Hence, \( N_i \) does not need to be large for checking the stability of the system (1) using Theorem 2.

Figure 1(b) shows that the stability boundary calculated by Theorem 2 is closer to the actual stability boundary than that by the Fridman’s theorem or Kolmanovskii’s theorem. It is shown that, through the example, our proposed method is less conservative than Fridman and Shaked (2002, 2003), and Kolmanovskii’s theorem in Kolmanovskii and Richard (1999). It is shown that, through the example, our proposed method is less conservative than Fridman and Shaked (2002, 2003), and Kolmanovskii and Richard (1999).

**Example 3:** Consider the second-order system with two delays

\[
x(t) = \begin{bmatrix} -3.0 & -2.5 \\ 1.0 & 0.0 \end{bmatrix} x(t) + \begin{bmatrix} 1.0 & -5.0 \\ 0.0 & -1.0 \end{bmatrix} x(t - h_1) + \begin{bmatrix} -1.0 & 0.5 \\ 0.0 & -1.0 \end{bmatrix} x(t - h_2)
\]

Figure 2 depicts the stable region as a function of time delays for Example 3 by using MATLAB LMI Toolbox (Gahinet et al. 1995). Figure 2 also shows that the stability boundary calculated by Theorem 2 is closer to the actual stability boundary than that by the Fridman’s theorem or Kolmanovskii’s theorem. It is shown that, through the example, our proposed method is less conservative than Fridman’s theorem and Kolmanovskii’s theorem.
Remark 5: Numerical examples show that small $N_i$ (or $N$) around 4 is enough to provide the improved results over the existing results. Also, for $N_i$ (or $N$) larger than 4, we do not get much improved results. Therefore, we may recommend $N_i = 4$ (or $N = 4$) to avoid the complexity of the LMI condition and obtain the improved results at the same time.

3.2. Neutral systems

Example 4: Consider the second order neutral system with two delays

Figure 3 depicts the stable region as a function of state delays, $h_1$ and $h_2$, for Example 4 by using MATLAB LMI Toolbox (Gahinet et al. 1995). From Figure 3(a), it can be seen that the stable region is expanded as $N$ is increased. It is shown that, through the example, the proposed method is less conservative than Fridman and Shaked (2003).

Example 5: Consider the second order neutral system with two delays

Figure 4 depicts the stable region as a function of state delays, $h_1$ and $h_2$, for Example 5 by using MATLAB LMI Toolbox (Gahinet et al. 1995). From this example, we show that the proposed method is less conservative than Fridman and Shaked (2003).

4. Conclusion

In this article, delay-dependent stability criteria for linear retarded and neutral systems with multiple delays are proposed by employing the Lyapunov-Krasovskii functional approach and integral inequality. Through numerical examples, it is shown that the proposed stability criteria are less
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References


