Ultimate Boundedness of Nonlinear Singularly Perturbed System with Measurement Noise

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SUMMARY In this letter, we consider the ultimate boundedness of the nonlinear singularly perturbed system with measurement noise. The composite controller is commonly used to regulate the singularly perturbed system. However, in the presence of measurement noise, the composite controller does not guarantee the ultimate boundedness of the singularly perturbed system. Thus, we propose the modified composite controller to show the ultimate boundedness of the singularly perturbed system with measurement noise.

key words: singular perturbation, measurement noise, ultimate boundedness

1. Introduction

In many practical systems, it is necessary to measure the state precisely to guarantee the control performances. Actually, however, there exists measurement noise that affects the control performance including the stability of the system. In the plethora of papers, although the uncertainties and time-delay are considered to show the robustness of the singularly perturbed systems [1], [2], it is still a challenging issue to show the stabilization or regulation of the singularly perturbed systems with measurement noise.

For the singularly perturbed system, there are some results of the robust stability analysis despite uncertainties or state time-delay. The studies of the robust stability in the existence of the uncertainties are investigated in [3] by using the norm bound information of the uncertainties and in [4] by introducing the quasi slow manifold and the contraction mapping. In [5], they deal with the small state time-delay on singularly perturbed systems, and in [6], the asymptotic stability bounds of the linear singularly perturbed system with time-delay is considered.

Recently, the results on measurement noise are appeared in [7]–[9]. To cope with the measurement noise, the second-order sliding-mode controller is proposed in [7] and the output feedback control using the high-gain observer is presented in [8] with switching-gain approach. In [9], the high-gain observer with the adaptive gain is suggested to improve the performance of the observer based on the space averaging technique.

To the authors’ best knowledge, there are no results about the singularly perturbed system with measurement noise. In this letter, we present the ultimate boundedness of the singularly perturbed system with measurement noise via the modified composite controller. As shown in [10] and [11], the measurement noise not only affects the system stability but also is critical to the estimation performance of the observer. To attenuate the destabilizing effect of the measurement noise, we propose the modified composite controller using the switching mechanism with the slow subsystem and the fast subsystem of the singularly perturbed system. Besides, an illustrative example is given to demonstrate our result.

2. Preliminaries

Consider the following nonlinear singularly perturbed system

\[
\begin{align*}
\dot{x} &= f_1(x,z) + g_1(x,z)u(x,z) \\
\epsilon \dot{z} &= f_2(x,z) + g_2(x,z)u(x,z)
\end{align*}
\]

where \( x \in D_x = \{ x \in \mathbb{R}^n : \| x \| \leq \rho_x \} \) and \( z \in D_z = \{ z \in \mathbb{R}^m : \| z \| \leq \rho_z \} \) are state vectors, \( f_i \) and \( g_i \) are smooth functions with \( f_i(0,0) = 0 \) and \( g_i(0,0) = 0 \) for \( i = 1, 2 \).

Assumption 1: The equation \( 0 = f_2(x,z) + g_2(x,z)u \) has an isolated root \( z = h(x,u) \).

To stabilize the above nonlinear singularly perturbed system (1), the slow controller \( u_s(x) \) is designed such that the slow subsystem \( \dot{x} = f_1(x,h(x,u_s)) + g_2(x,h(x,u_s))u_s \) is exponentially stable and the fast controller \( u_f(x,z) \) is designed such that

(i) \( u_f(x,h(x,u_s)) = 0 \),
(ii) \( f_2(x,z) + g_2(x,z)[u_s(x) + u_f(x,z)] = 0 \) has an isolated root \( z = h(x,u_s) \),
(iii) the fast subsystem \( \dot{y} = f_2(x,y + h(x,u_s)) + g_2(x,y + h(x,u_s))[u_s + u_f] \) is uniformly exponentially stable.

Then, the composite controller \( \tilde{u} = u_s(x) + u_f(x,z) \) makes the nonlinear singularly perturbed system exponentially stable [1].

3. Main Results

The presence of measurement noise is inevitable in many practical systems. In this respect, we consider the nonlinear singularly perturbed system with measurement noise as follows:

\[
\begin{align*}
\dot{x} &= f_1(x,z) + g_1(x,z)u(x) \\
\epsilon \dot{z} &= f_2(x,z) + g_2(x,z)u(x)
\end{align*}
\]
where $\chi = \begin{bmatrix} x_1 \; x_2 \end{bmatrix}^T = X + \mu$, $X = \begin{bmatrix} x^T \; z^T \end{bmatrix}^T$ and $\mu$ is the measurement noise with $\mu = \begin{bmatrix} \mu^T \; \mu^T \end{bmatrix}^T$. From the design procedure of the composite controller as shown in the preliminaries, there exists a Lyapunov function $V(x)$ for the slow subsystem such that
\begin{equation}
\frac{b_1 \|x\|^2 \leq V(x) \leq b_2 \|x\|^2}{\frac{\partial V}{\partial x} \left\{ f_1(x, h(x, u_x)) \right\} + g_1(x, h(x, u_x))u_x} \leq -b_3 \|x\|^2 \right) \tag{3}
\end{equation}

and there exists a Lyapunov function $W(x, y)$ such that
\begin{equation}
\frac{c_1 \|y\|^2 \leq W(x, y) \leq c_2 \|y\|^2}{\frac{\partial W}{\partial y} \left\{ f_2(x, y + h(x, u_x)) \right\} + g_2(x, y + h(x, u_x))(u_x + u_y)} \leq -c_3 \|y\|^2 \right) \tag{4}
\end{equation}

Now, we consider the ultimate boundedness of the nonlinear singularly perturbed system with measurement noise (2).

**Theorem 1:** Consider the nonlinear singularly perturbed system with measurement noise (2). Assume that there exists a composite controller $\bar{u} = u_1 + u_2$ such that the nonlinear singularly perturbed system (1) is exponentially stable. Then, there exist $e^*$ and $\mu^*$ such that for all $\epsilon < e^*$ with $\|\mu\| \leq \mu^*$, the singularly perturbed system with measurement noise (2) is ultimately bounded by the following modified composite controller:
\begin{equation}
u = \bar{u}(x_1, x_2) - u_1(x_1, x_2)\text{sign} \left( \frac{\partial W}{\partial y}g_2(x_1, x_2) \right) \tag{5}
\end{equation}

where $u_1(x_1, x_2) = \gamma_{u1} \|x\|$ with a positive constant $\gamma_{u1}$ and 
\[\text{sign}(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ -1 & \text{if } s < 0 \end{cases}\]

**Proof:** The closed-loop system of the nonlinear singularly perturbed system with measurement noise and the modified composite controller is
\begin{equation}
\begin{cases}
\dot{x} = f_1(x, z) + g_1(x, z)u(\chi)
\dot{\chi} = f_2(x, z) + g_2(x, z)u(\chi)
\end{cases} \tag{6}
\end{equation}

with $u(\chi) = \bar{u}(x_1, x_2) - u_1(x_1, x_2)\text{sign} \left( \frac{\partial W}{\partial y}g_2(x_1, x_2) \right)$, where $\chi = \begin{bmatrix} x_1 \; x_2 \end{bmatrix}^T$, $\chi_1 = x + \mu$, and $\chi_2 = z + \mu$. To show the ultimate boundedness, we set the Lyapunov function $v = V(x) + W(x, y)$. Then, the time-derivative of the Lyapunov function $v$ along the closed-loop system (6) is
\begin{equation}
\dot{v} = \frac{\partial V}{\partial x} \left\{ f_1(x, z) - f_1(x, h(x, u_x)) \right\} + \frac{\partial W}{\partial y} \left\{ f_1(x, h(x, u_x)) + g_1(x, h(x, u_x))u_x \right\}
+ \frac{\partial W}{\partial y} \left\{ f_1(x, h(x, u_x)) \right\} \left\{ u_j(x, z) - u_j(x, h(x, u_x)) \right\}
+ \frac{\partial W}{\partial y} \left\{ f_1(x, z) - g_1(x, h(x, u_x)) \right\} \bar{u}(X)
\end{equation}

where $\text{sign}(\chi) = \text{sign} \left( \frac{\partial W}{\partial y}g_2(x_1, x_2) \right)$. From the continuously differentiable functions, $f_1$ and $g_1$, there exist positive constants, $k_{f1}$ and $k_{g1}$ such that
\begin{align}
\|f_1(x, z) - f_1(x, z)\| &\leq k_{f1} \|x - z\| + k_{g1} \|z - \bar{z}\| \tag{8}
\|g_1(x, z) - g_1(x, z)\| &\leq k_{g1} \|x - z\| + k_{g1} \|z - \bar{z}\| \tag{9}
\end{align}

By using the above inequalities, we rewrite the time-derivative of the Lyapunov function $v$ as
\begin{align}
\dot{v} &\leq - \frac{\|x\|^T}{\|y\|} Q \left[ \frac{\|x\|}{\|y\|} \right] + \frac{\partial V}{\partial x} \left\{ f_1(x, z) \right\} + \frac{\partial W}{\partial y} \left\{ f_1(x, h(x, u_x)) \right\}
+ \frac{\partial W}{\partial y} \left\{ f_1(x, h(x, u_x)) \right\} \bar{u}(X)
+ \frac{\partial W}{\partial y} \left\{ f_1(x, z) - f_1(x, h(x, u_x)) \right\} \bar{u}(X)
+ \frac{\partial W}{\partial y} \left\{ f_1(x, z) - g_1(x, h(x, u_x)) \right\} \bar{u}(X)
\end{align}

where $Q = \begin{bmatrix} b_3 & -\alpha \\ -\alpha & c_3 - \left( c_s k_{f1} \rho_y + c_s k_{g1} \right) \end{bmatrix}$

with $\alpha = \frac{1}{2} \left[ b_4 (k_{f1} + k_{g1} \rho_y) + c_s k_{g1} (1 + k_h) (k_{f1} + k_{g1} \rho_y) + c_s k_{g1} k_{h1} \rho_y^2 + c_s k_{h1} (1 + k_h) (k_{f1} + k_{g1} \rho_y) + c_s k_{h1} k_{g1} \rho_y k_h \right]$ and $\rho_y = \rho_1 + k_{g1} \rho_y^n$. Now, we show there exist $\gamma_0$ and $\mu^*$ such that the sign of $\frac{\partial W}{\partial y}g_2(x, z)$ is the same with that of $\frac{\partial W}{\partial y}g_2(x_1, x_2)$ in $\|x\| \geq \gamma_0 \|y\|$ with $\|\mu\| \leq \mu^*$. First, $\text{sign}(\chi)$ is rewritten as
\begin{equation}
\text{sign}(\chi) = \text{sign} \left( \frac{\partial W}{\partial y}g_2(x_1, x_2) \right) \tag{10}
\end{equation}

From the smooth function $g_2$, there exists $k_1 > 0$ such that
\begin{equation}
\left\| \frac{\partial W}{\partial y}g_2(x) - \frac{\partial W}{\partial y}g_2(X) \right\| \leq k_1 \|y\| \tag{11}
\end{equation}

in $D_x \times D_x$. Therefore, from the above inequality, there exist $\gamma_0$ and $\mu^*$ such that
\[
\text{sign} \left( \frac{\partial W}{\partial y} g_2(X) + k_i \mu^* \right) = \text{sign} \left( \frac{\partial W}{\partial y} g_2(X) \right)
\]

(14)

From (12)–(14), we obtain
\[
\text{sign} \left( \frac{\partial W}{\partial y} g_2(x) \right) = \text{sign} \left( \frac{\partial W}{\partial y} g_2(X) \right)
\]

(15)
in \( \|X\| \geq \gamma_0 \|\mu\| \) with \( \|\mu\| \leq \mu^* \). Then, we have the time-derivative of the Lyapunov function \( \nu \) as
\[
\dot{\nu} \leq -\left[ \frac{\|X\|}{\|y\|} \right]^T Q \left[ \frac{\|X\|}{\|y\|} \right] + \frac{1}{\epsilon} \left\| \frac{\partial W}{\partial y} g_2(X) \right\|
\]

\[
\cdot \left\| \|u(x) - \bar{u}(X)\| - u_1(\chi) \right\| + O(\epsilon) \left\| \|\mu\| \right\|
\]

(16)

for \( \|X\| \geq \gamma_0 \|\mu\| \) with \( \mu^* \). From the above time-derivative of the Lyapunov function, there exists \( \epsilon_2^* > 0 \) such that the matrix \( Q \) is positive-definite. Note that \( \|X\| \geq (1 + \gamma_1) \|\mu\| \) where \( \gamma_1 \) is a positive constant implies \( \|\mu\| \leq \gamma_1^{-1} \|X\| \). From \( u_1(\chi) = \gamma_1 \|\chi\| \) and the above inequality, there exists \( \gamma_1 \) that satisfies the following inequality.
\[
\|u(x) - \bar{u}(X)\| \leq k_\mu \|\mu\| \leq k_\mu \gamma_1^{-1} \|\chi\| \leq u_1(\chi)
\]

(17)

Thus, there exist \( \epsilon_1^* > 0 \) and \( \gamma_1 > 0 \) such that the second term of the time-derivative of the Lyapunov function is negative in \( \|X\| \geq (1 + \gamma_1) \|\mu\| \). Then, we have
\[
\dot{\nu} \leq -\alpha_2(\|X\|)
\]

(19)

\[
\dot{\nu} \leq -\alpha_2(\|X\|)
\]

(20)

for all \( \|X\| \geq \bar{\mu} = \max(\gamma_0, 1 + \gamma_1) \|\mu\| \) where \( \alpha_i \) are class \( K \) functions for \( i = 1, 2 \). Then, take \( r > 0 \) such that \( B_r \subset D_r \ni X \subset D \) where \( D_r = \{\|X\| < r\} \) with \( i = \bar{\mu}, r \). Let \( \Gamma = B_r \setminus B_\bar{\mu} \) and \( k = \min \alpha_2(X) \). Therefore, we have \( \dot{\nu} \leq -k \)

\[
\nu(t) \leq \nu(X(t)) \leq k(t) - k(t_0) \leq c - k(t - t_0)
\]

(21)

where \( \nu(X) \leq c \) for all \( X \in \Gamma \). Therefore, \( \nu(X(t)) \) reduces to \( \bar{\mu} \) within the time interval \([t_0, t_0 + (c - \bar{\mu})/k]\), and the ultimate bound is obtained as
\[
\nu(X(t)) \leq \bar{\mu} \quad \Rightarrow \quad \alpha_1(\|X(t)\|) \leq \bar{\mu}
\]

Therefore, we show that \( \|X(t)\| \leq \alpha_1^{-1}(\bar{\mu}) \).

4. Illustrative Example

Consider the electric op-amp circuit in Fig. 1. Apply the Kirchhoff current law to \( v_1, v_2 \) with \( R_2 = R_3 = R_4 = 1 \Omega \) and obtain the state equation as
\[
\dot{v}_1 = -\frac{1}{R_2 C_1} v_1 + \frac{1}{R_1 C_1} v_2
\]

Fig. 1 Electric Op-Amp circuit [12].

Fig. 2 Performance with the conventional controller: (a) state response, (b) input with \( \gamma_{\text{con}} = 0 \).

\[
C_2 \dot{v}_2 = \frac{1}{R_1} v_1 + \left(1 - \frac{1}{R_1}\right) v_2 - u
\]

(22)

Define \( \alpha_1 = x \) and \( \alpha_2 = z \) with \( R_1 = 0.5 \Omega, C_1 = 1 \text{F}, C_2 = 0.1 \text{F} \), then we have
\[
\dot{x} = -2x + 2z
\]

\[
\dot{z} = 2x - z - u
\]

(23)

where \( \epsilon = 0.1 \). First, we design the composite controller \( \bar{u} = u_s + u_f \) through the preliminaries where \( u_s(t) = 1.1 x(t) \) and \( u_f(t) = -0.9(\bar{z}(t) - 0.9 x(t)) \) with \( V(x) = 1/2 x^2 \) and \( W(y) = 1/2 y^2 \) \((y = z - 0.9 x) \). Then, we propose the modified composite controller as
\[
\dot{u}(\chi) = \bar{u}(\chi_1, \chi_2) - u_1(\chi_1, \chi_2) \text{sign}(0.9 \chi_1 - \chi_2)
\]

(24)

where \( u_1(\chi_1, \chi_2) = \gamma_{\text{con}} \sqrt{\chi_1^2 + \chi_2^2} \) with \( \gamma_{\text{con}} = 5 \). From Theorem 1 with the modified composite controller (24), the singularly perturbed system with the measurement noise is ultimately bounded as shown in Fig. 3. Moreover, in the presence of the measurement noise, the proposed method in Fig. 3 shows the better performance compared with the conventional composite controller in Fig. 2. We use the white gaussian measurement noise whose variance is set to 0.5 in the simulation.
5. Conclusions

We have considered the ultimate boundedness of the nonlinear singularly perturbed system with measurement noise. With the existence of the measurement noise, it is impossible to guarantee the boundedness of the nonlinear singularly perturbed system by the composite controller. Thus, we show the ultimate boundedness of the singularly perturbed system with the measurement noise via the modified composite controller. The contribution of our proposed method is shown through the simulation example.

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